B. Sc Part III PAPER – V

M_n — test for the uniform convergence of a sequence

Let $\{f_n\}$ be a sequence of functions such that $\lim_{n\to\infty} f_n(x) = f(x)\ \forall x \in [a,b]$

Let $M_n = \sup \{f_n(x) - f(x)\}$.

Then $f_n \to f$ uniformly on $[a,b]$ iff $M_n \to 0$ as $n \to \infty$

The Condition is necessary:

Let the sequence $\{f_n(x)\}$ converge uniformly to $f(x)$ in $a \le x \le b$.

Then for every $\varepsilon > 0$ there exists a positive integer m, independent of x such that

$|f_n(x) - f(x)| < \varepsilon$

for every $n \geq m$ and for every $x \in [a,b]$

$\Rightarrow M_n = \sup |f_n(x) - f(x)| < \varepsilon$ for $n \geq m$

i.e. $M_n < \varepsilon$ for $n \geq m$, i.e.

$M_n \to 0$ as $n \to \infty$

Page No.:
Date: / /

The Condition is Sufficient

We shall prove that if $M_n \to 0$ as $n \to \infty$, then the Sequence

$\{f_n(x)\}$ is uniformly convergent in $[a,b]$

Now $M_n \to 0 \Rightarrow \sup |f_n(x) - f(x)| \to 0$

$\Rightarrow |f_n(x) - f(x)| \to 0$

Thus $|f_n(x) - f(x)| < M_n < \epsilon$

for all $x \in [a,b]$

Hence the Sequence $\{f_n(x)\}$ is uniformly convergent in $[a,b]$